Understanding Sparse JL for Feature Hashing

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NeurIPS 2019 (Poster #59)

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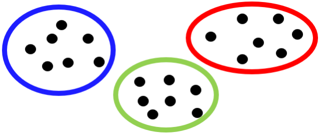

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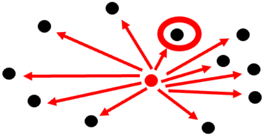

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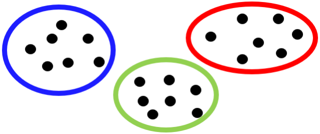

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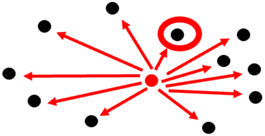

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Key question: What is the tradeoff between the dimension m, the performance in distance preservation, and the projection time?

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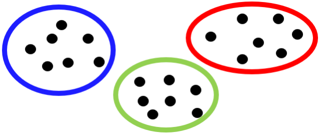

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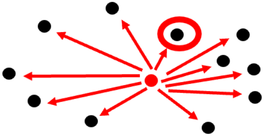

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Key question: What is the tradeoff between the dimension m, the performance in distance preservation, and the projection time?

This paper: A theoretical analysis of this tradeoff for a state-of-the-art dimensionality reduction scheme on feature vectors.

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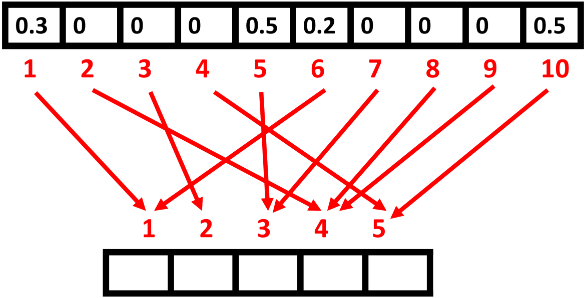

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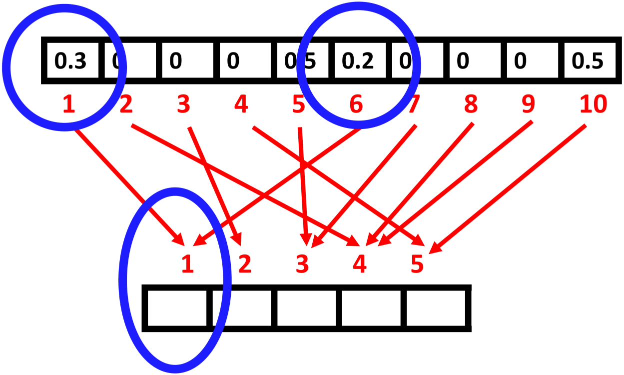

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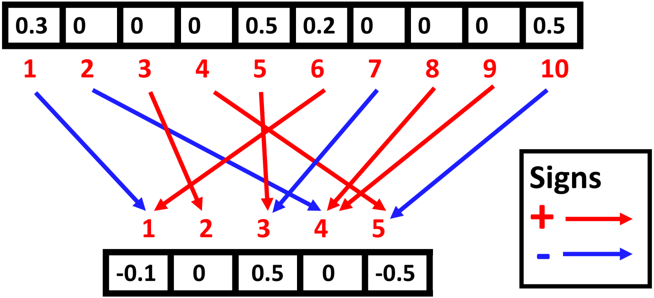

Use **random signs** to handle collisions: $f(x)_i = \sum_{j \in h^{-1}(i)} \sigma_j x_j$.

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This work

Analysis of tradeoff for sparse JL between # of hash functions s, dimension m, and performance in ℓ_2-distance preservation.

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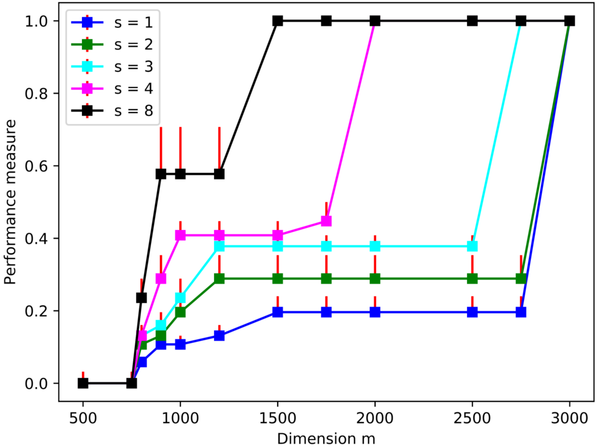

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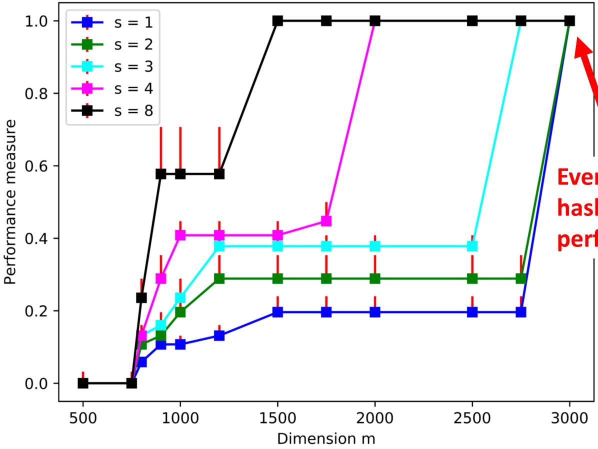

Even feature hashing has full performance

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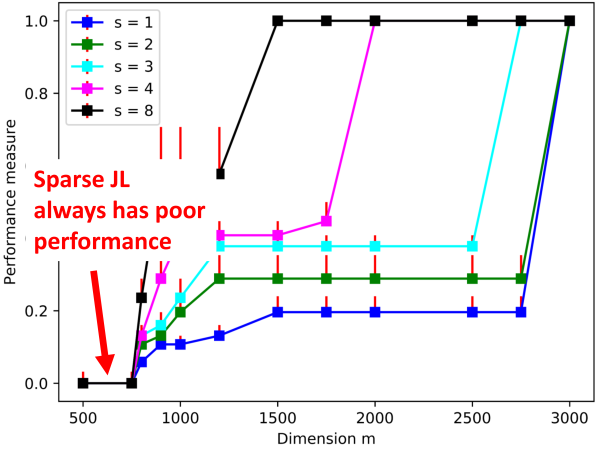

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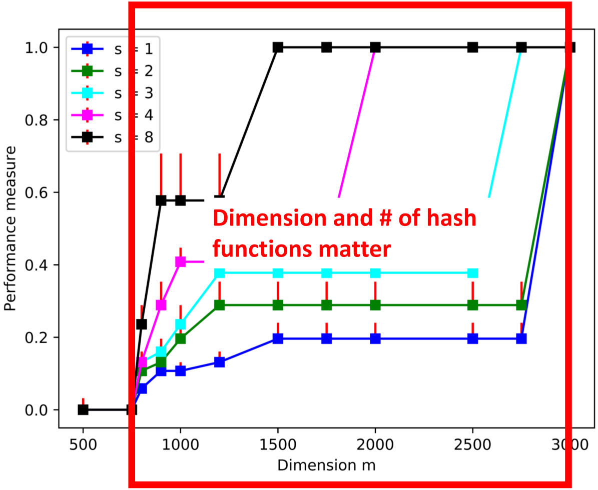

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Sparse JL can sometimes perform much better in practice on feature vectors than traditional theory suggests...

Performance on feature vectors (Weinberger et al. '09)

Consider vectors w/ small ℓ_∞-to-ℓ_2 norm ratio:

$$S_v = \left\{x \in \mathbb{R}^n \mid \|x\|_\infty \leq v \|x\|_2\right\}.$$

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$v(m, \epsilon, \delta, s)$ is the supremum over $v \in [0, 1]$ such that:

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- $v(m, \epsilon, \delta, s) = 0 \implies$ poor performance
- $v(m, \epsilon, \delta, s) = 1 \implies$ full performance
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We give a tight theoretical analysis of the function $v(m, \epsilon, \delta, s)$.

Informal statement of main result

Goal: $\mathbb{P}_{f \in \mathcal{F}}[\|f(x)\|_2 \in (1 \pm \epsilon) \|x\|_2] > 1 - \delta$.

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Theorem (Informal)

For error ϵ and failure probability δ, sparse JL with projected dimension m and s hash functions has **four regimes** *in its performance: that is,*

$$v(m,\epsilon,\delta,s) = \begin{cases} 1 \text{ (full performance)} & \text{High } m \\ \sqrt{s}B_1 \text{ (partial performance)} & \text{Middle } m \\ \sqrt{s}\min(B_1,B_2) \text{ (partial performance)} & \text{Middle } m \\ 0 \text{ (poor performance)} & \text{Small } m, \end{cases}$$

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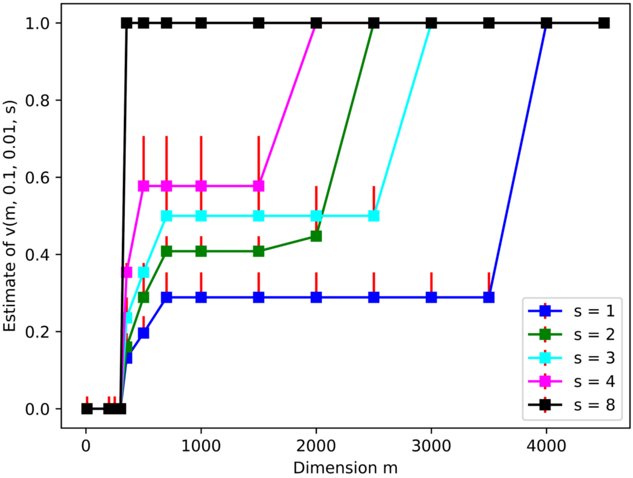
$v(m, \epsilon, \delta, s)$ on more synthetic data



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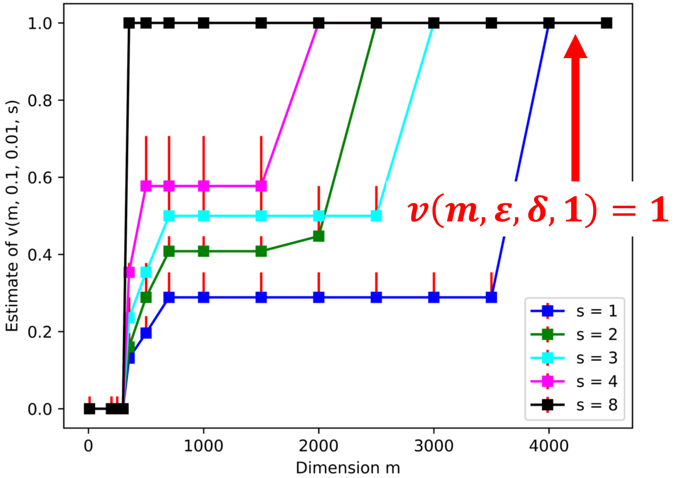

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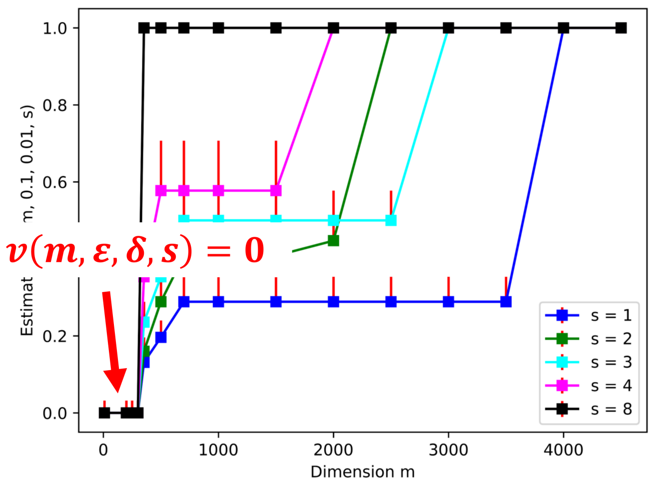

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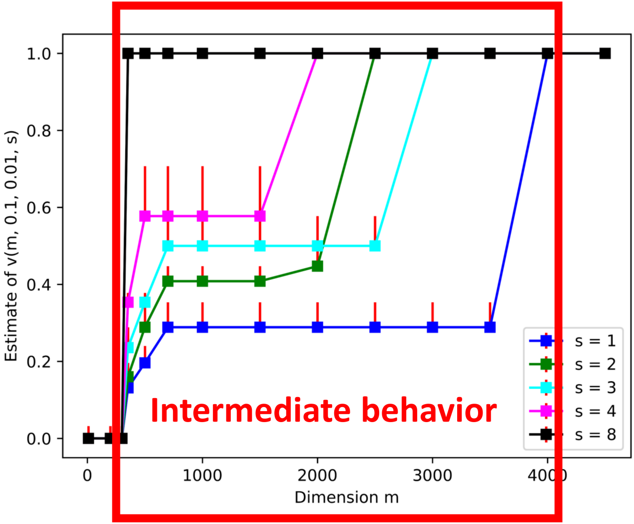
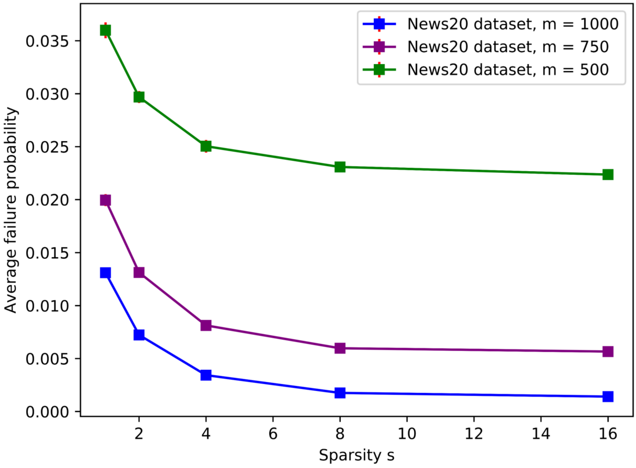


Sparse JL on News20 dataset



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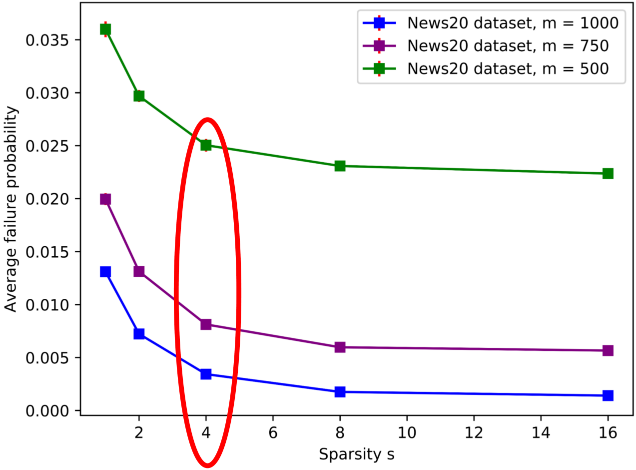

Sparse JL with 4 hash fns can significantly outperform feature hashing!

Comparison to previous work

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Bounds on v (Weinberger et al. '09,..., Freksen et al. '18):

- $v(m, \epsilon, \delta, \mathbf{1})$ understood
- $v(m, \epsilon, \delta, s)$ bound for *multiple hashing* (a suboptimal construction)

Bounds for sparse JL on full space $\mathbb{R}^n$:

- Can set $m \approx \epsilon^{-2} \log(1/\delta)$, $s \approx \epsilon^{-1} \log(1/\delta)$ (Kane and Nelson '12)
- Can set $m \approx \min(2\epsilon^{-2}/\delta, \epsilon^{-2} \log(1/\delta) e^{\Theta(\epsilon^{-1} \log(1/\delta)/s)})$ (Cohen '16)

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$\implies$ *Characterization of sparse JL performance in terms of* ϵ, δ, *and* ℓ_∞*-to-*ℓ_2 *norm ratio for a general # of hash functions* s

Conclusion

Tight analysis of $v(m, \epsilon, \delta, s)$ for uniform sparse JL for a general s. Could inform how to optimally set s and m in practice.

Characterization of sparse JL performance in terms of ϵ, δ, and ℓ_∞-to-ℓ_2 norm ratio for a general # of hash functions s.

Evaluation on real-world and synthetic data (sparse JL can perform much better than feature hashing).

Proof technique involves a new perspective on analyzing JL distributions.

Thank you!